

# COLORFUL SUBHYPERGRAPHS IN UNIFORM HYPERGRAPHS

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**ABSTRACT.** There are several topological results ensuring the existence of a large complete bipartite subgraph in any properly colored graph satisfying some special topological regularity conditions. In view of  $\mathbb{Z}_p$ -Tucker lemma, Alishahi and Hajiabolhassan [*On the chromatic number of general Kneser hypergraphs*, *Journal of Combinatorial Theory, Series B*, 2015] introduced a lower bound for the chromatic number of Kneser hypergraphs  $\text{KG}^r(\mathcal{H})$ . Next, Meunier [*Colorful subhypergraphs in Kneser hypergraphs*, *The Electronic Journal of Combinatorics*, 2014] improved their result by proving that any properly colored general Kneser hypergraph  $\text{KG}^r(\mathcal{H})$  contains a large colorful  $r$ -partite subhypergraph provided that  $r$  is prime. In this paper, we give some new generalizations of  $\mathbb{Z}_p$ -Tucker lemma. Hence, improving Meunier's result in some aspects. Some new lower bounds for the chromatic number and local chromatic number of uniform hypergraphs are presented as well.

**Keyword:** chromatic number of hypergraphs,  $\mathbb{Z}_p$ -Tucker-Ky Fan lemma, colorful complete hypergraph,  $\mathbb{Z}_p$ -box-complex,  $\mathbb{Z}_p$ -hom-complex

## 1. Introduction

**1.1. Background and Motivations.** In 1955, Kneser [18] posed a conjecture about the chromatic number of Kneser graphs. In 1978, Lovász [20] proved this conjecture by using algebraic topology. The Lovász's proof marked the beginning of the history of topological combinatorics. Nowadays, it is an active stream of research to study the coloring properties of graphs by using algebraic topology. There are several lower bounds for the chromatic number of graphs related to the indices of some topological spaces defined based on the structure of graphs. However, for hypergraphs, there are a few such lower bounds, see [6, 11, 17, 19, 27].

A *hypergraph*  $\mathcal{H}$  is a pair  $(V(\mathcal{H}), E(\mathcal{H}))$ , where  $V(\mathcal{H})$  is a finite set, called the vertex set of  $\mathcal{H}$ , and  $E(\mathcal{H})$  is a family of nonempty subsets of  $V(\mathcal{H})$ , called the edge set of  $\mathcal{H}$ . Throughout the paper, by a nonempty hypergraph, we mean a hypergraph with at least one edge. If any edge  $e \in E(\mathcal{H})$  has the cardinality  $r$ , then the hypergraph  $\mathcal{H}$  is called  *$r$ -uniform*. For a set  $U \subseteq V(\mathcal{H})$ , the *induced subhypergraph on  $U$* , denoted  $\mathcal{H}[U]$ , is a hypergraph with the vertex set  $U$  and the edge set  $\{e \in E(\mathcal{H}) : e \subseteq U\}$ . Throughout the paper, by a *graph*, we mean a 2-uniform hypergraph. Let  $r \geq 2$  be a positive integer and  $q \geq r$  be an integer. An  $r$ -uniform hypergraph  $\mathcal{H}$  is called  *$q$ -partite* with parts  $V_1, \dots, V_q$  if

- $V(\mathcal{H}) = \bigcup_{i=1}^q V_i$  and
- each edge of  $\mathcal{H}$  intersects each part  $V_i$  in at most one vertex.

If  $\mathcal{H}$  contains all possible edges, then we call it a *complete  $r$ -uniform  $q$ -partite hypergraph*. Also, we say the hypergraph  $\mathcal{H}$  is *balanced* if the values of  $|V_j|$  for  $j = 1, \dots, q$  differ by at most one, i.e.,  $|V_i| - |V_j| \leq 1$  for each  $i, j \in [q]$ .

Let  $\mathcal{H}$  be an  $r$ -uniform hypergraph and  $U_1, \dots, U_q$  be  $q$  pairwise disjoint subsets of  $V(\mathcal{H})$ . The hypergraph  $\mathcal{H}[U_1, \dots, U_q]$  is a subhypergraph of  $\mathcal{H}$  with the vertex set  $\bigcup_{i=1}^q U_i$  and the edge set

$$E(\mathcal{H}[U_1, \dots, U_q]) = \left\{ e \in E(\mathcal{H}) : e \subseteq \bigcup_{i=1}^q U_i \text{ and } |e \cap U_i| \leq 1 \text{ for each } i \in [q] \right\}.$$

Note that  $\mathcal{H}[U_1, \dots, U_q]$  is an  $r$ -uniform  $q$ -partite hypergraph with parts  $U_1, \dots, U_q$ . By the symbol  $\binom{[n]}{r}$ , we mean the family of all  $r$ -subsets of the set  $[n]$ . The hypergraph  $K_n^r = ([n], \binom{[n]}{r})$  is called the complete  $r$ -uniform hypergraph with  $n$  vertices. For  $r = 2$ , we would rather use  $K_n$  instead of  $K_n^2$ . The largest possible integer  $n$  such that  $\mathcal{H}$  contains  $K_n^r$  as a subhypergraph is called the *clique number* of  $\mathcal{H}$ , denoted  $\omega(\mathcal{H})$ .

A *proper  $t$ -coloring* of a hypergraph  $\mathcal{H}$  is a map  $c : V(\mathcal{H}) \rightarrow [t]$  such that there is no monochromatic edge. The minimum possible such a  $t$  is called the *chromatic number* of  $\mathcal{H}$ , denoted  $\chi(\mathcal{H})$ . If there is no such a  $t$ , we define the chromatic number to be infinite. Let  $c$  be a proper coloring of  $\mathcal{H}$  and  $U_1, \dots, U_q$  be  $q$  pairwise disjoint subsets of  $V(\mathcal{H})$ . The hypergraph  $\mathcal{H}[U_1, \dots, U_q]$  is said to be *colorful* if for each  $j \in [q]$ , the vertices of  $U_j$  get pairwise distinct colors. For a properly colored graph  $G$ , a subgraph is called *multicolored* if its vertices get pairwise distinct colors.

For a hypergraph  $\mathcal{H}$ , the *Kneser hypergraph*  $\text{KG}^r(\mathcal{H})$  is an  $r$ -uniform hypergraph with the vertex set  $E(\mathcal{H})$  and whose edges are formed by  $r$  pairwise vertex-disjoint edges of  $\mathcal{H}$ , i.e.,

$$E(\text{KG}^r(\mathcal{H})) = \{ \{e_1, \dots, e_r\} : e_i \cap e_j = \emptyset \text{ for each } i \neq j \in [r] \}.$$

For any graph  $G$ , it is known that there are several hypergraphs  $\mathcal{H}$  such that  $\text{KG}^2(\mathcal{H})$  and  $G$  are isomorphic. The Kneser hypergraph  $\text{KG}^r(K_n^k)$  is called the “usual” Kneser hypergraph which is denoted by  $\text{KG}^r(n, k)$ . Coloring properties of Kneser hypergraphs have been studied extensively in the literature. Lovász [20] (for  $r = 2$ ) and Alon, Frankl and Lovász [7] determined the chromatic number of  $\text{KG}^r(n, k)$ . For an integer  $r \geq 2$ , they proved that

$$\chi(\text{KG}^r(n, k)) = \left\lceil \frac{n - r(k - 1)}{r - 1} \right\rceil.$$

For a hypergraph  $\mathcal{H}$ , the  $r$ -colorability defect of  $\mathcal{H}$ , denoted  $\text{cd}_r(\mathcal{H})$ , is the minimum number of vertices which should be removed such that the induced hypergraph on the remaining vertices is  $r$ -colorable, i.e.,

$$\text{cd}_r(\mathcal{H}) = \min \{ |U| : \mathcal{H}[V(\mathcal{H}) \setminus U] \text{ is } r\text{-colorable} \}.$$

For a hypergraph  $\mathcal{H}$ , Dol’nikov [11] (for  $r = 2$ ) and Kříž [19] proved that

$$\chi(\text{KG}^r(\mathcal{H})) \geq \left\lceil \frac{\text{cd}_r(\mathcal{H})}{r - 1} \right\rceil,$$

which is a generalization of the results by Lovász [20] and Alon, Frankl and Lovász [7].

For a positive integer  $r$ , let  $\mathbb{Z}_r = \{\omega, \omega^2, \dots, \omega^r\}$  be a cyclic group of order  $r$  with generator  $\omega$ . Consider a vector  $X = (x_1, x_2, \dots, x_n) \in (\mathbb{Z}_r \cup \{0\})^n$ . An alternating subsequence of  $X$  is a sequence  $x_{i_1}, x_{i_2}, \dots, x_{i_m}$  of nonzero terms of  $X$  such that  $i_1 < \dots < i_m$  and  $x_{i_j} \neq x_{i_{j+1}}$  for each  $j \in [m - 1]$ . We denote by  $\text{alt}(x)$  the maximum possible length of an alternating subsequence of  $X$ . For a vector  $X = (x_1, x_2, \dots, x_n) \in (\mathbb{Z}_r \cup \{0\})^n$  and for an  $\epsilon \in \mathbb{Z}_p$ , set  $X^\epsilon = \{i \in [n] : x_i = \epsilon\}$ . Note that, by abuse of notation, we can write  $X = (X^\epsilon)_{\epsilon \in \mathbb{Z}_r}$ . For two vectors  $X, Y \in (\mathbb{Z}_r \cup \{0\})^n$ , by  $X \subseteq Y$ , we mean  $X^\epsilon \subseteq Y^\epsilon$  for each  $\epsilon \in \mathbb{Z}_r$ .

For a hypergraph  $\mathcal{H}$  and a bijection  $\sigma : [n] \rightarrow V(\mathcal{H})$ , define

$$\text{alt}_r(\mathcal{H}, \sigma) = \max \{ \text{alt}(X) : X \in (\mathbb{Z}_r \cup \{0\})^n \text{ such that } E(\mathcal{H}[\sigma(X^\epsilon)]) = \emptyset \text{ for each } \epsilon \in \mathbb{Z}_r \}.$$

Also, let

$$\text{alt}_r(\mathcal{H}) = \min_{\sigma} \text{alt}_r(\mathcal{H}, \sigma),$$

where the minimum is taken over all bijection  $\sigma : [n] \rightarrow V(\mathcal{H})$ . One can readily check that for any hypergraph  $\mathcal{H}$ ,  $|V(\mathcal{H})| - \text{alt}_r(\mathcal{H}) \geq \text{cd}_r(\mathcal{H})$  and the inequality is often strict, see [6]. Alishahi and Hajiabolhassan [6] improved Dol'nikov-Křiz result by proving that for any hypergraph  $\mathcal{H}$  and for any integer  $r \geq 2$ , the quantity  $\left\lceil \frac{|V(\mathcal{H})| - \text{alt}_r(\mathcal{H})}{r-1} \right\rceil$  is a lower bound for the chromatic number of  $\text{KG}^r(\mathcal{H})$ . Using this lower bound, the chromatic number of some families of graphs and hypergraphs are computed, see [1, 2, 3, 4, 6, 14]. There are some other lower bounds for the chromatic number of graphs which are better than the former discussed lower bounds. They are based on some topological indices of some topological spaces connected to the structure of graphs. In spite of these lower bounds being better, they are not combinatorial and most of the times they are difficult to compute.

The existence of large colorful bipartite subgraphs in a properly colored graph has been extensively studied in the literature [6, 8, 9, 26, 27, 28]. To be more specific, there are several theorems ensuring the existence of a colorful bipartite subgraph in any properly colored graph such that the bipartite subgraph has a specific number of vertices related to some topological parameters connected to the graph. Simonyi and Tardos [28] improved Dol'nikov's lower bound and proved that in any proper coloring of a Kneser graph  $\text{KG}^2(\mathcal{H})$ , there is a multicolored complete bipartite graph  $K_{\lfloor \frac{\text{cd}_2(\mathcal{H})}{2} \rfloor, \lfloor \frac{\text{cd}_2(\mathcal{H})}{2} \rfloor}$  such that the  $\text{cd}^2(\mathcal{H})$  different colors occur alternating on the two parts of the bipartite graph with respect to their natural order. By a combinatorial proof, Alishahi and Hajiabolhassan [6] improved this result. They proved that the result remains true if we replace  $\text{cd}^2(\mathcal{H})$  by  $n - \text{alt}_2(\mathcal{H})$ . Also, a stronger result is proved by Simonyi, Tardif, and Zsbán [26].

**Theorem A.** (Zig-zag Theorem [26]). *Let  $G$  be a nonempty graph which is properly colored with arbitrary number of colors. Then  $G$  contains a multicolored complete bipartite subgraph  $K_{\lfloor \frac{t}{2} \rfloor, \lfloor \frac{t}{2} \rfloor}$ , where  $\text{Xind}(\text{Hom}(K_2, G)) + 2 = t$ . Moreover, colors appear alternating on the two sides of the bipartite subgraph with respect to their natural ordering.*

The quantity  $\text{Xind}(\text{Hom}(K_2, G))$  is the cross-index of hom-complex  $\text{Hom}(K_2, G)$  which will be defined in Subsection 2.2. We should mention that there are some other weaker similar results in terms of some other topological parameters, see [27, 28].

Note that prior mentioned results concern the existence of colorful bipartite subgraphs in properly colored graphs (2-uniform hypergraphs). In 2014, Meunier [23] found the first colorful type result for the uniform hypergraphs. He proved that for any prime number  $p$ , any properly colored Kneser hypergraph  $\text{KG}^p(\mathcal{H})$  must contain a colorful balanced complete  $p$ -uniform  $p$ -partite subhypergraph with a specific number of vertices, see Theorem C.

**1.2. Main Results.** For a given graph  $G$ , there are several complexes defined based on the structure of  $G$ . For instance, the box-complex of  $G$ , denoted  $\text{B}_0(G)$ , and the hom-complex of  $G$ , denoted  $\text{Hom}(K_2, G)$ , see [21, 26, 27]. Also, there are some lower bounds for the chromatic number of graphs related to some indices of these complexes [26, 27]. In this paper, we naturally generalize the definitions of box-complex and hom-complex of graphs to uniform hypergraphs. Also, the definition of  $\mathbb{Z}_p$ -cross-index of  $\mathbb{Z}_p$ -posets will be introduced. Using these complexes, as a first main result of this paper, we generalize Meunier's result [23] (Theorem C) to the following theorem.

**Theorem 1.** *Let  $r \geq 2$  be a positive integer and  $p \geq r$  be a prime number. Assume that  $\mathcal{H}$  is an  $r$ -uniform hypergraph and  $c : V(\mathcal{H}) \rightarrow [C]$  is a proper coloring of  $\mathcal{H}$  ( $C$  arbitrary). Then we have the following assertions.*

- (i) *There is some colorful balanced complete  $r$ -uniform  $p$ -partite subhypergraph in  $\mathcal{H}$  with  $\text{ind}_{\mathbb{Z}_p}(B_0(\mathcal{H}, \mathbb{Z}_p)) + 1$  vertices. In particular,*

$$\chi(\mathcal{H}) \geq \frac{\text{ind}_{\mathbb{Z}_p}(B_0(\mathcal{H}, \mathbb{Z}_p)) + 1}{r - 1}.$$

- (ii) *If  $p \leq \omega(\mathcal{H})$ , then there is some colorful balanced complete  $r$ -uniform  $p$ -partite subhypergraph in  $\mathcal{H}$  with  $\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^r, \mathcal{H})) + p$  vertices. In particular,*

$$\chi(\mathcal{H}) \geq \frac{\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^r, \mathcal{H})) + p}{r - 1}.$$

Quantities  $\text{ind}_{\mathbb{Z}_p}(B_0(\mathcal{H}, \mathbb{Z}_p))$  and  $\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^r, \mathcal{H}))$  appearing in the statement of Theorem 1 are respectively the  $\mathbb{Z}_p$ -index and  $\mathbb{Z}_p$ -cross-index of the  $\mathbb{Z}_p$ -box-complex  $B_0(\mathcal{H}, \mathbb{Z}_p)$  and  $\mathbb{Z}_p$ -hom-complex  $\text{Hom}(K_p^r, \mathcal{H})$  which will be defined in Subsection 2.2. Using these complexes, we introduce some new lower bounds for the chromatic number of uniform hypergraphs. In view of Theorem 1, next theorem provides a hierarchy of lower bounds for the chromatic number of  $r$ -uniform hypergraphs.

**Theorem 2.** *Let  $r \geq 2$  be a positive integer and  $p \geq r$  be a prime number. For any  $r$ -uniform hypergraph  $\mathcal{H}$ , we have the following inequalities.*

- (i) *If  $p \leq \omega(\mathcal{H})$ , then*

$$\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^r, \mathcal{H})) + p \geq \text{ind}_{\mathbb{Z}_p}(B_0(\mathcal{H}, \mathbb{Z}_p)) + 1.$$

- (ii) *If  $\mathcal{H} = \text{KG}^r(\mathcal{F})$  for some hypergraph  $\mathcal{F}$ , then*

$$\text{ind}_{\mathbb{Z}_p}(B_0(\mathcal{H}, \mathbb{Z}_p)) + 1 \geq |V(\mathcal{F})| - \text{alt}_p(\mathcal{F}) \geq \text{cd}_p(\mathcal{F}).$$

In view of Theorem 2, Theorem 1 is a common extension of Theorem A and Theorem C. Furthermore, for  $r = 2$ , Theorem 1 implies the next corollary which also is a generalization of Theorem A.

**Corollary 1.** *Let  $p$  be a prime number and let  $G$  be a nonempty graph which is properly colored with arbitrary number of colors. Then there is a multicolored complete  $p$ -partite subgraph  $K_{n_1, n_2, \dots, n_p}$  of  $G$  such that*

- $\sum_{i=1}^p n_i = \text{ind}_{\mathbb{Z}_p}(B_0(G, \mathbb{Z}_p)) + 1$ ,
- $|n_i - n_j| \leq 1$  for each  $i, j \in [p]$ .

Moreover, if  $p \leq \omega(G)$ , then  $\text{ind}_{\mathbb{Z}_p}(B_0(G, \mathbb{Z}_p)) + 1$  can be replaced with  $\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p, G)) + p$ .

In view of the prior mentioned results, the following question naturally arises.

**Question 1.** Do Theorem 1 and Theorem 2 remain true for non-prime  $p$ ?

**1.3. Applications to Local Chromatic Number of Uniform Hypergraphs.** For a graph  $G$  and a vertex  $v \in V(G)$ , the *closed neighborhood of  $v$* , denoted  $N[v]$ , is the set  $\{v\} \cup \{u : uv \in E(G)\}$ . The *local chromatic number* of  $G$ , denoted  $\chi_l(G)$ , is defined in [12] as follows:

$$\chi_l(G) = \min_c \max\{|c(N[v])| : v \in V(G)\}$$

where the minimum is taken over all proper coloring  $c$  of  $G$ . Note that Theorem A gives the following lower bound for the local chromatic number of a nonempty graph  $G$ :

$$(1) \quad \chi_l(G) \geq \left\lceil \frac{\text{Xind}(\text{Hom}(K_2, G)) + 2}{2} \right\rceil + 1.$$

Note that for a Kneser hypergraph  $\text{KG}^2(\mathcal{H})$ , by using Simonyi and Tardos colorful result [28] or the extension given by Alishahi and Hajiabolhassan [6], there are two similar lower bounds for

$\chi_l(\text{KG}^2(\mathcal{H}))$  which respectively used  $\text{cd}_2(\mathcal{H})$  and  $|V(\mathcal{H})| - \text{alt}_2(\mathcal{H})$  instead of  $\text{Xind}(\text{Hom}(K_2, G)) + 2$ . However, as it is stated in Theorem 2, the lower bound in terms of  $\text{Xind}(\text{Hom}(K_2, G)) + 2$  is better than these two last mentioned lower bounds. Using Corollary 1, we have the following lower bound for the local chromatic number of graphs.

**Corollary 2.** *Let  $G$  be a nonempty graph and  $p$  be a prime number. Then*

$$\chi_l(G) \geq t - \left\lfloor \frac{t}{p} \right\rfloor + 1,$$

where  $t = \text{ind}_{\mathbb{Z}_p}(B_0(G, \mathbb{Z}_p)) + 1$ . Moreover, if  $p \leq \omega(G)$ , then  $\text{ind}_{\mathbb{Z}_p}(B_0(G, \mathbb{Z}_p)) + 1$  can be replaced with  $\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p, G)) + p$ .

Note that if we set  $p = 2$ , then previous theorem implies Simonyi and Tardos lower bound for the local chromatic number. Note that, in general, this lower bound might be better than Simonyi and Tardos lower bound. To see this, let  $k \geq 2$  be a fixed integer. Consider the Kneser graph  $\text{KG}^2(n, k)$  and let  $p = p(n)$  be a prime number such that  $p = O(\ln n)$ . By Theorem 2, for  $n \geq pk$ , we have

$$\text{ind}_{\mathbb{Z}_p}(B_0(\text{KG}^2(n, k), \mathbb{Z}_p)) + 1 \geq \text{cd}_p(K_n^k) = n - p(k - 1).$$

Note that the lower bound for  $\chi_l(\text{KG}^2(n, k))$  coming from Inequality 1 is

$$(2) \quad 1 + \left\lceil \frac{n - 2k + 2}{2} \right\rceil = \frac{n}{2} - o(1),$$

while, in view of Corollary 2, we have

$$\chi_l(\text{KG}^2(n, k)) \geq n - p(k - 1) - \left\lfloor \frac{n - p(k - 1)}{p} \right\rfloor + 1 = n - o(n),$$

which is better than the quantity in Equation 2 if  $n$  is sufficiently large. However, since the induced subgraph on the neighbors of any vertex of  $\text{KG}(n, k)$  is isomorphic to  $\text{KG}(n - k, k)$ , we have

$$\chi_l(\text{KG}(n, k)) \geq n - 3(k - 1).$$

**Corollary 3.** *Let  $\mathcal{F}$  be a hypergraph and  $\alpha(\mathcal{F})$  be its independence number. Then for any prime number  $p$ , we have*

$$\chi_l(\text{KG}^2(\mathcal{F})) \geq \left\lceil \frac{(p - 1)|V(\mathcal{F})|}{p} \right\rceil - (p - 1) \cdot \alpha(\mathcal{F}) + 1.$$

*Proof.* In view of Theorem 2, we have

$$\text{ind}_{\mathbb{Z}_p}(B_0(\text{KG}^2(\mathcal{F}), \mathbb{Z}_p)) + 1 \geq \text{cd}_p(\mathcal{F}) \geq |V(\mathcal{F})| - p \cdot \alpha(\mathcal{F}).$$

Now, Corollary 2 implies the assertion.  $\square$

Meunier [23] naturally generalized the definition of local chromatic number of graphs to uniform hypergraphs as follows. Let  $\mathcal{H}$  be a uniform hypergraph. For a set  $X \subseteq V(\mathcal{H})$ , the closed neighborhood of  $X$  is the set  $X \cup \mathcal{N}(X)$ , where

$$\mathcal{N}(X) = \{v \in V(\mathcal{H}) : \exists e \in E(\mathcal{H}) \text{ such that } e \setminus X = \{v\}\}.$$

For a uniform hypergraph  $\mathcal{H}$ , the local chromatic number of  $\mathcal{H}$  is defined as follows:

$$\chi_l(\mathcal{H}) = \min_c \max\{|c(\mathcal{N}[e \setminus \{v\}])| : e \in E(\mathcal{H}) \text{ and } v \in e\},$$

where the minimum is taken over all proper coloring  $c$  of  $\mathcal{H}$ .

Meunier [23], by using his colorful theorem (Theorem C), generalized Simonyi and Tardos lower bound [28] for the local chromatic number of Kneser graphs to the local chromatic number of Kneser hypergraphs. He proved:

$$\chi_l(\text{KG}^p(\mathcal{H})) \geq \min \left( \left\lceil \frac{|V(\mathcal{H})| - \text{alt}_p(\mathcal{H})}{p} \right\rceil + 1, \left\lceil \frac{|V(\mathcal{H})| - \text{alt}_p(\mathcal{H})}{p-1} \right\rceil \right)$$

for any hypergraph  $\mathcal{H}$  and any prime number  $p$ . In what follows, we generalize this result.

**Theorem 3.** *Let  $\mathcal{H}$  be an  $r$ -uniform hypergraph with at least one edge and  $p$  be a prime number, where  $r \leq p \leq \omega(\mathcal{H})$ . Let  $t = \text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^r, \mathcal{H})) + p$ . If  $t = ap + b$ , where  $a$  and  $b$  are nonnegative integers and  $0 \leq b \leq p-1$ , then*

$$\chi_l(\mathcal{H}) \geq \min \left( \left\lceil \frac{(p-r+1)a + \min\{p-r+1, b\}}{r-1} \right\rceil + 1, \left\lceil \frac{t}{r-1} \right\rceil \right).$$

*Proof.* Let  $c$  be an arbitrary proper coloring of  $\mathcal{H}$  and let  $\mathcal{H}[U_1, \dots, U_p]$  be the colorful balanced complete  $r$ -uniform  $p$ -partite subhypergraph of  $\mathcal{H}$  whose existence is ensured by Theorem 1. Note that  $b$  numbers of  $U_i$ 's, say  $U_1, \dots, U_b$ , have the cardinality  $\lceil \frac{t}{r} \rceil$  while the others have the cardinality  $\lfloor \frac{t}{r} \rfloor \geq 1$ . Consider  $U_1, \dots, U_{p-r+1}$ . Two different cases will be distinguished.

**Case 1.** If  $\left| \bigcup_{i=1}^{p-r+1} c(U_i) \right| < \left\lceil \frac{t}{r-1} \right\rceil$ , then there is a vertex  $v \in \bigcup_{i=p-r+2}^p U_i$  whose color is not in  $\bigcup_{i=1}^{p-r+1} c(U_i)$ . Consider an edge of  $\mathcal{H}[U_1, \dots, U_p]$  containing  $v$  and such that  $|e \cap U_{p-r+1}| = 1$  and  $e \cap U_i = \emptyset$  for  $i = 1, \dots, p-r$ . Let  $e \cap U_{p-r+1} = \{u\}$ . One can check that

$$\{c(v)\} \cup \left( \bigcup_{i=1}^{p-r+1} c(U_i) \right) \subseteq c(\mathcal{N}(e \setminus \{u\})).$$

Therefore, since any color is appeared in at most  $r-1$  number of  $U_i$ 's, we have

$$\left| \bigcup_{i=1}^{p-r+1} c(U_i) \right| \geq \left\lceil \frac{\sum_{i=1}^{p-r+1} |U_i|}{r-1} \right\rceil,$$

and consequently,

$$|c(\mathcal{N}(e \setminus \{u\}))| \geq 1 + \left\lceil \frac{\sum_{i=1}^{p-r+1} |U_i|}{r-1} \right\rceil = 1 + \left\lceil \frac{(p-r+1)a + \min\{p-r+1, b\}}{r-1} \right\rceil,$$

which completes the proof in Case 1.

**Case 2.** If  $\left| \bigcup_{i=1}^{p-r+1} c(U_i) \right| \geq \left\lceil \frac{t}{r-1} \right\rceil$ , then consider an edge of  $\mathcal{H}[U_1, \dots, U_p]$  such that  $|e \cap U_{p-r+1}| = 1$  and  $e \cap U_i = \emptyset$  for  $i = 1, \dots, p-r$ . Let  $e \cap U_{p-r+1} = \{u\}$ . One can see that

$$\bigcup_{i=1}^{p-r+1} c(U_i) \subseteq c(\mathcal{N}(e \setminus \{u\})),$$

which completes the proof in Case 2. □



**Corollary 4.** *Let  $\mathcal{H}$  be a  $p$ -uniform hypergraph with at least one edge, where  $p$  is a prime number. Then*

$$\chi_l(\mathcal{H}) \geq \min \left( \left\lceil \frac{\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^p, \mathcal{H})) + p}{p} \right\rceil + 1, \left\lceil \frac{\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^p, \mathcal{H})) + p}{p-1} \right\rceil \right).$$

*Proof.* Since  $\mathcal{H}$  has at least one edge, we have  $\omega(\mathcal{H}) \geq p$ . Therefore, in view of Theorem 3, we have the assertion.  $\square$

Note that if  $\mathcal{H} = \text{KG}^p(\mathcal{F})$ , then, in view of Theorem 2, we have

$$\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^p, \mathcal{H})) + p \geq |V(\mathcal{F})| - \text{alt}_p(\mathcal{F}).$$

This implies that the previous corollary is a generalization of Meunier's lower bound for the local chromatic number of  $\text{KG}^p(\mathcal{F})$

**1.4. Plan.** Section 2 contains some backgrounds and essential definitions used elsewhere in the paper. In Section 3, we present some new topological tools which help us for the proofs of main results. Section 4 is devoted to the proofs of Theorem 1 and Theorem 2.

## 2. Preliminaries

**2.1. Topological Indices and Lower Bound for Chromatic Number.** We assume basic knowledge in combinatorial algebraic topology. Here, we are going to bring a brief review of some essential notations and definitions which will be needed throughout the paper. For more, one can see the book written by Matoušek [21]. Also, the definitions of box-complex, hom-complex, and cross-index will be generalized to  $\mathbb{Z}_p$ -box-complex,  $\mathbb{Z}_p$ -hom-complex, and  $\mathbb{Z}_p$ -cross-index, respectively.

Let  $\mathbb{G}$  be a finite nontrivial group which acts on a topological space  $X$ . We call  $X$  a *topological  $\mathbb{G}$ -space* if for each  $g \in \mathbb{G}$ , the map  $g : X \rightarrow X$  which  $x \mapsto g \cdot x$  is continuous. A *free topological  $\mathbb{G}$ -space*  $X$  is a topological  $\mathbb{G}$ -space such that  $\mathbb{G}$  acts on it freely, i.e., for each  $g \in \mathbb{G} \setminus \{e\}$ , the map  $g : X \rightarrow X$  has no fixed point. For two topological  $\mathbb{G}$ -spaces  $X$  and  $Y$ , a continuous map  $f : X \rightarrow Y$  is called a  $\mathbb{G}$ -map if  $f(g \cdot x) = g \cdot f(x)$  for each  $g \in \mathbb{G}$  and  $x \in X$ . We write  $X \xrightarrow{\mathbb{G}} Y$  to mention that there is a  $\mathbb{G}$ -map from  $X$  to  $Y$ . A map  $f : X \rightarrow Y$  is called a  $\mathbb{G}$ -equivariant map, if  $f(g \cdot x) = g \cdot f(x)$  for each  $g \in \mathbb{G}$  and  $x \in X$ .

Simplicial complexes provide a bridge between combinatorics and topology. A simplicial complex can be viewed as a combinatorial object, called abstract simplicial complex, or as a topological space, called geometric simplicial complex. Here, we just remind the definition of an abstract simplicial complex. However, we assume that the reader is familiar with the concept of how an abstract simplicial complex and its geometric realization are connected to each other. A *simplicial complex* is a pair  $(V, K)$ , where  $V$  is a finite set and  $K$  is a family of subsets of  $V$  such that if  $F \in K$  and  $F' \subseteq F$ , then  $F' \in K$ . Any set in  $K$  is called a simplex. Since we may assume that  $V = \bigcup_{F \in K} F$ , we can write  $K$  instead of  $(V, K)$ . The *dimension* of  $K$  is defined as follows:

$$\dim(K) = \max\{|F| - 1 : F \in K\}.$$

The geometric realization of  $K$  is denoted by  $\|K\|$ . For two simplicial complexes  $C$  and  $K$ , by a *simplicial map*  $f : C \rightarrow K$ , we mean a map from  $V(C)$  to  $V(K)$  such that the image of any simplex of  $C$  is a simplex of  $K$ . For a nontrivial finite group  $\mathbb{G}$ , a *simplicial  $\mathbb{G}$ -complex*  $K$  is a simplicial complex with a  $\mathbb{G}$ -action on its vertices such that each  $g \in \mathbb{G}$  induces a simplicial map from  $K$  to  $K$ , that is the map which maps  $v$  to  $g \cdot v$  for each  $v \in V(K)$ . If for each  $g \in \mathbb{G} \setminus \{e\}$ , there is no fixed simplex under the simplicial map made by  $g$ , then  $K$  is called a *free simplicial  $\mathbb{G}$ -complex*. For a simplicial  $\mathbb{G}$ -complex  $K$ , if we take the affine extension, then  $K$  is free if and only if  $\|K\|$  is free. For two simplicial  $\mathbb{G}$ -complexes  $C$  and  $K$ , a simplicial map  $f : C \rightarrow K$  is called a

simplicial  $\mathbb{G}$ -map if  $f(g \cdot v) = g \cdot f(v)$  for each  $g \in \mathbb{G}$  and  $v \in V(C)$ . We write  $C \xrightarrow{\mathbb{G}} K$ , if there is a simplicial  $\mathbb{G}$ -map from  $C$  to  $K$ . Note that if  $C \xrightarrow{\mathbb{G}} K$ , then  $\|C\| \xrightarrow{\mathbb{G}} \|K\|$ . A map  $f : C \rightarrow K$  is called a  $\mathbb{G}$ -equivariant map, if  $f(g \cdot v) = g \cdot f(v)$  for each  $g \in \mathbb{G}$  and  $v \in V(C)$ .

For an integer  $n \geq 0$  and a nontrivial finite group  $\mathbb{G}$ ,  $E_n\mathbb{G}$  space is a free  $(n-1)$ -connected  $n$ -dimensional simplicial  $\mathbb{G}$ -complexes. A concrete example of an  $E_n\mathbb{G}$  space is the  $(n+1)$ -fold join  $\mathbb{G}^{*(n+1)}$ . As a topological space  $\mathbb{G}^{*(n+1)}$  is a  $(n+1)$ -fold join of an  $(n+1)$ -point discrete space. This is known that for any two  $E_n\mathbb{G}$  space  $X$  and  $Y$ , there is a  $\mathbb{G}$ -map from  $X$  to  $Y$ .

For a  $\mathbb{G}$ -space  $X$ , define

$$\text{ind}_{\mathbb{G}}(X) = \min\{n : X \xrightarrow{\mathbb{G}} E_n\mathbb{G}\}.$$

Note that here  $E_n\mathbb{G}$  can be any  $E_n\mathbb{G}$ , since there is a  $\mathbb{G}$ -map between any two  $E_n\mathbb{G}$  spaces, see [21]. Also, for a simplicial complex  $K$ , by  $\text{ind}_{\mathbb{G}}(K)$ , we mean  $\text{ind}_{\mathbb{G}}(\|K\|)$ . Throughout the paper, for  $\mathbb{G} = \mathbb{Z}_2$ , we would rather use  $\text{ind}(-)$  instead of  $\text{ind}_{\mathbb{Z}_2}(-)$ .

**Properties of the  $\mathbb{G}$ -index.** [21] Let  $\mathbb{G}$  be a finite nontrivial group.

- (i)  $\text{ind}_{\mathbb{G}}(X) > \text{ind}_{\mathbb{G}}(Y)$  implies  $X \not\xrightarrow{\mathbb{G}} Y$ .
- (ii)  $\text{ind}_{\mathbb{G}}(E_n\mathbb{G}) = n$  for any  $E_n\mathbb{G}$  space.
- (iii)  $\text{ind}_{\mathbb{G}}(X * Y) \leq \text{ind}_{\mathbb{G}}(X) + \text{ind}_{\mathbb{G}}(Y) + 1$ .
- (iv) If  $X$  is  $(n-1)$ -connected, then  $\text{ind}_{\mathbb{G}}(X) \geq n$ .
- (v) If  $K$  is a free simplicial  $\mathbb{G}$ -complex of dimension  $n$ , then  $\text{ind}_{\mathbb{G}}(K) \leq n$ .

**2.2.  $\mathbb{Z}_p$ -Box-Complex,  $\mathbb{Z}_p$ -Poset, and  $\mathbb{Z}_p$ -Hom-Complex.** In this subsection, for any  $r$ -uniform hypergraph  $\mathcal{H}$ , we are going to define two objects;  $\mathbb{Z}_p$ -box-complex of  $\mathcal{H}$  and  $\mathbb{Z}_p$ -hom-complex of  $\mathcal{H}$  which the first one is a simplicial  $\mathbb{Z}_p$ -complex and the second one is a  $\mathbb{Z}_p$ -poset. Moreover, for any  $\mathbb{Z}_p$ -poset  $P$ , we assign a combinatorial index to  $P$  called the cross-index of  $P$ .

**$\mathbb{Z}_p$ -Box-Complex.** Let  $r \geq 2$  be a positive integer and  $p \geq r$  be a prime number. For an  $r$ -uniform hypergraph  $\mathcal{H}$ , define the  $\mathbb{Z}_p$ -box-complex of  $\mathcal{H}$ , denoted  $B_0(\mathcal{H}, \mathbb{Z}_p)$ , to be a simplicial complex with the vertex set  $\biguplus_{i=1}^p V(\mathcal{H}) = \mathbb{Z}_p \times V(\mathcal{H})$  and the simplex set consisting of all  $\{\omega^1\} \times U_1 \cup \dots \cup \{\omega^p\} \times U_p$ ,

where

- $U_1, \dots, U_p$  are pairwise disjoint subsets of  $V(\mathcal{H})$ ,
- $\bigcup_{i=1}^p U_i \neq \emptyset$ , and
- the hypergraph  $\mathcal{H}[U_1, U_2, \dots, U_p]$  is a complete  $r$ -uniform  $p$ -partite hypergraph.

Note that some of  $U_i$ 's might be empty. In fact, if  $U_1, \dots, U_p$  are pairwise disjoint subsets of  $V(\mathcal{H})$  and the number of nonempty  $U_i$ 's is less than  $r$ , then  $\mathcal{H}[U_1, U_2, \dots, U_p]$  is a complete  $r$ -uniform  $p$ -partite hypergraph and thus  $\{\omega^1\} \times U_1 \cup \dots \cup \{\omega^p\} \times U_p \in B_0(\mathcal{H}, \mathbb{Z}_p)$ . For each  $\epsilon \in \mathbb{Z}_p$  and each  $(\epsilon', v) \in V(B_0(\mathcal{H}, \mathbb{Z}_p))$ , define  $\epsilon \cdot (\epsilon', v) = (\epsilon \cdot \epsilon', v)$ . One can see that this action makes  $B_0(\mathcal{H}, \mathbb{Z}_p)$  a free simplicial  $\mathbb{Z}_p$ -complex. It should be mentioned that the  $\mathbb{Z}_2$ -box-complex  $B_0(\mathcal{H}, \mathbb{Z}_2)$  is extensively studied in the literature, see [27, 28]. In the literature, for a graph  $G$ , the simplicial complex  $B_0(G, \mathbb{Z}_2)$  is shown by  $B_0(G)$ . This simplicial complex is used to introduce some lower bounds for the chromatic number of a given graph  $G$ , see [27]. In particular, we have the following inequalities

$$\chi(G) \geq \text{ind}(B_0(G)) + 1 \geq \text{coind}(B_0(G)) + 1 \geq n - \text{alt}(\mathcal{F}) \geq \text{cd}_2(\mathcal{F}),$$

where  $\mathcal{F}$  is any hypergraph such that  $\text{KG}^2(\mathcal{F})$  and  $G$  are isomorphic, see [2, 6, 27].



**$\mathbb{Z}_p$ -Poset.** A partially ordered set, or simply a *poset*, is defined as an ordered pair  $P = (V(P), \preceq)$ , where  $V(P)$  is a set called the ground set of  $P$  and  $\preceq$  is a partial order on  $V(P)$ . For two posets  $P$  and  $Q$ , by an order-preserving map  $\phi : P \rightarrow Q$ , we mean a map  $\phi$  from  $V(P)$  to  $V(Q)$  such that for each  $u, v \in V(P)$ , if  $u \preceq v$ , then  $\phi(u) \preceq \phi(v)$ . A poset  $P$  is called a  $\mathbb{Z}_p$ -poset, if  $\mathbb{Z}_p$  acts on  $V(P)$  and furthermore, for each  $\epsilon \in \mathbb{Z}_p$ , the map  $\epsilon : V(P) \rightarrow V(P)$  which  $v \mapsto \epsilon \cdot v$  is an automorphism of  $P$  (order preserving bijective map). If for each  $\epsilon \in \mathbb{Z}_p \setminus \{e\}$ , this map has no fixed point, then  $P$  is called a *free  $\mathbb{Z}_p$ -poset*. For two  $\mathbb{Z}_p$ -poset  $P$  and  $Q$ , by an order-preserving  $\mathbb{Z}_p$ -map  $\phi : P \rightarrow Q$ , we mean an order-preserving map from  $V(P)$  to  $V(Q)$  such that for each  $v \in V(P)$  and  $\epsilon \in \mathbb{Z}_p$ , we have  $\phi(\epsilon \cdot v) = \epsilon \cdot \phi(v)$ . If there exists such a map, we write  $P \xrightarrow{\mathbb{Z}_p} Q$ .

For a nonnegative integer  $n$  and a prime number  $p$ , let  $Q_{n,p}$  be a free  $\mathbb{Z}_p$ -poset with ground set  $\mathbb{Z}_p \times [n+1]$  such that for any two members  $(\epsilon, i), (\epsilon', j) \in Q_{n,p}$ ,  $(\epsilon, i) <_{Q_{n,p}} (\epsilon', j)$  if  $i < j$ . Clearly,  $Q_{n,p}$  is a free  $\mathbb{Z}_p$ -poset with the action  $\epsilon \cdot (\epsilon', j) = (\epsilon \cdot \epsilon', j)$  for each  $\epsilon \in \mathbb{Z}_p$  and  $(\epsilon', j) \in Q_{n,p}$ . For a  $\mathbb{Z}_p$ -poset  $P$ , the  $\mathbb{Z}_p$ -cross-index of  $P$ , denoted  $\text{Xind}_{\mathbb{Z}_p}(P)$ , is the least integer  $n$  such that there is a  $\mathbb{Z}_p$ -map from  $P$  to  $Q_{n,p}$ . Throughout the paper, for  $p = 2$ , we speak about  $\text{Xind}(-)$  rather than  $\text{Xind}_{\mathbb{Z}_2}(-)$ . It should be mentioned that  $\text{Xind}(-)$  is first defined in [26].

Let  $P$  be a poset. We can define an *order complex*  $\Delta P$  with the vertex set same as the ground set of  $P$  and simplex set consisting of all chains in  $P$ . One can see that if  $P$  is a free  $\mathbb{Z}_p$ -poset, then  $\Delta P$  is a free simplicial  $\mathbb{Z}_p$ -complex. Moreover, any order-preserving  $\mathbb{Z}_p$ -map  $\phi : P \rightarrow Q$  can be lifted to a simplicial  $\mathbb{Z}_p$ -map from  $\Delta P$  to  $\Delta Q$ . Clearly, there is a simplicial  $\mathbb{Z}_p$ -map from  $\Delta Q_{n,p}$  to  $\mathbb{Z}_p^{*(n+1)}$  (identity map). Therefore, if  $\text{Xind}_{\mathbb{Z}_p}(P) = n$ , then we have a simplicial  $\mathbb{Z}_p$ -map from  $\Delta P$  to  $\mathbb{Z}_p^{*(n+1)}$ . This implies that  $\text{Xind}_{\mathbb{Z}_p}(P) \geq \text{ind}_{\mathbb{Z}_p}(\Delta P)$ . Throughout the paper, for each  $(\epsilon, j) \in Q_{n,p}$ , when we speak about the sign of  $(\epsilon, j)$  and the absolute value of  $(\epsilon, j)$ , we mean  $\epsilon$  and  $j$ , respectively.

**Theorem B.** [5] *Let  $P$  be a free  $\mathbb{Z}_2$ -poset and  $\phi : P \rightarrow Q_{s,2}$  be an order preserving  $\mathbb{Z}_2$ -map. Then  $P$  contains a chain  $p_1 \prec_P \cdots \prec_P p_k$  such that  $k = \text{Xind}(P) + 1$  and the signs of  $\phi(p_i)$  and  $\phi(p_{i+1})$  differ for each  $i \in [k-1]$ . Moreover, if  $s = \text{Xind}(P)$ , then for any  $(s+1)$ -tuple  $(\epsilon_1, \dots, \epsilon_{s+1}) \in \mathbb{Z}_2^{s+1}$ , there is at least one chain  $p_1 \prec_P \cdots \prec_P p_{s+1}$  such that  $\phi(p_i) = (\epsilon_i, i)$  for each  $i \in [s+1]$ .*

**$\mathbb{Z}_p$ -Hom-Complex.** Let  $\mathcal{H}$  be an  $r$ -uniform hypergraph. Also, let  $p \geq r$  be a prime number. The  $\mathbb{Z}_p$ -hom-complex  $\text{Hom}(K_p^r, \mathcal{H})$  is a free  $\mathbb{Z}_p$ -poset with the ground set consisting of all ordered  $p$ -tuples  $(U_1, \dots, U_p)$ , where  $U_i$ 's are nonempty pairwise disjoint subsets of  $V$  and  $\mathcal{H}[U_1, \dots, U_p]$  is a complete  $r$ -uniform  $p$ -partite hypergraph. For two  $p$ -tuples  $(U_1, \dots, U_p)$  and  $(U'_1, \dots, U'_p)$  in  $\text{Hom}(K_p^r, \mathcal{H})$ , we define  $(U_1, \dots, U_p) \preceq (U'_1, \dots, U'_p)$  if  $U_i \subseteq U'_i$  for each  $i \in [p]$ . Also, for each  $\omega^i \in \mathbb{Z}_p = \{\omega^1, \dots, \omega^p\}$ , let  $\omega^i \cdot (U_1, \dots, U_p) = (U_{1+i}, \dots, U_{p+i})$ , where  $U_j = U_{j-p}$  for  $j > p$ . Clearly, this action is a free  $\mathbb{Z}_p$ -action on  $\text{Hom}(K_p^r, \mathcal{H})$ . Consequently,  $\text{Hom}(K_p^r, \mathcal{H})$  is a free  $\mathbb{Z}_p$ -poset with this  $\mathbb{Z}_p$ -action.

For a nonempty graph  $G$  and for  $p = 2$ , it is proved [2, 6, 26, 27] that

$$(3) \quad \begin{aligned} \chi(G) &\geq \text{Xind}(\text{Hom}(K_2, G)) + 2 \geq \text{ind}(\Delta \text{Hom}(K_2, G)) + 2 \geq \text{ind}(B_0(G)) + 1 \\ &\geq \text{coind}(B_0(G)) + 1 \geq |V(\mathcal{F})| - \text{alt}_2(\mathcal{F}) \geq \text{cd}_2(\mathcal{F}), \end{aligned}$$

where  $\mathcal{F}$  is any hypergraph such that  $\text{KG}^2(\mathcal{F})$  and  $G$  are isomorphic.

### 3. Notations and Tools

For a simplicial complex  $K$ , by  $\text{sd } K$ , we mean the first barycentric subdivision of  $K$ . It is the simplicial complex whose vertex set is the set of nonempty simplices of  $K$  and whose simplices are the collections of simplices of  $K$  which are pairwise comparable by inclusion. Throughout the paper, by  $\sigma_{t-1}^{r-1}$ , we mean the  $(t-1)$ -dimensional simplicial complex with vertex set  $\mathbb{Z}_r$  containing

all  $t$ -subsets of  $\mathbb{Z}_r$  as its maximal simplices. The join of two simplicial complexes  $C$  and  $K$ , denoted  $C * K$ , is a simplicial complex with the vertex set  $V(C) \uplus V(K)$  and such that the set of its simplices is  $\{F_1 \uplus F_2 : F_1 \in C \text{ and } F_2 \in K\}$ . Clearly, we can see  $\mathbb{Z}_r$  as a 0-dimensional simplicial complex. Note that the vertex set of simplicial complex  $\text{sd} \mathbb{Z}_r^{*\alpha}$  can be identified with  $(\mathbb{Z}_r \cup \{0\})^\alpha \setminus \{\mathbf{0}\}$  and the vertex set of  $(\sigma_{t-1}^{r-1})^{*n}$  is the set of all pairs  $(\epsilon, i)$ , where  $\epsilon \in \mathbb{Z}_r$  and  $i \in [n]$ .

**3.1.  $\mathbb{Z}_p$ -Tucker-Ky Fan lemma.** The famous Borsuk-Ulam theorem has many generalizations which have been extensively used in investigating graph coloring properties. Some of these interesting generalizations are Tucker lemma [29],  $\mathbb{Z}_p$ -Tucker Lemma [30], and Tucker-Ky Fan [13]. For more details about the Borsuk-Ulam theorem and its generalizations, we refer the reader to [21].

Actually, Tucker lemma is a combinatorial counterpart of Borsuk-Ulam theorem. There are several interesting and surprising applications of Tucker Lemma in combinatorics, including a combinatorial proof of Lovász-Kneser theorem by Matoušek [22].

**Lemma A.** (Tucker lemma [29]). *Let  $m$  and  $n$  be positive integers and  $\lambda : \{-1, 0, +1\}^n \setminus \{(0, \dots, 0)\} \rightarrow \{\pm 1, \pm 2, \dots, \pm m\}$  be a map satisfying the following properties:*

- *for any  $X \in \{-1, 0, +1\}^n \setminus \{\mathbf{0}\}$ , we have  $\lambda(-X) = -\lambda(X)$  (a  $\mathbb{Z}_2$ -equivariant map),*
- *no two signed vectors  $X$  and  $Y$  are such that  $X \subseteq Y$  and  $\lambda(X) = -\lambda(Y)$ .*

*Then, we have  $m \geq n$ .*

Another interesting generalization of the Borsuk-Ulam theorem is Ky Fan's lemma [13]. This generalization ensures that with the same assumptions as in Lemma A, there is odd number of chains  $X_1 \subseteq X_2 \subseteq \dots \subseteq X_n$  such that

$$\{\lambda(X_1), \dots, \lambda(X_n)\} = \{+c_1, -c_2, \dots, (-1)^{n-1}c_n\},$$

where  $1 \leq c_1 < \dots < c_n \leq m$ . Ky Fan's lemma has been used in several articles to study some coloring properties of graphs, see [5, 9, 15]. There are also some other generalizations of Tucker Lemma. A  $\mathbb{Z}_p$  version of Tucker Lemma, called  $\mathbb{Z}_p$ -Tucker Lemma, is proved by Ziegler [30] and extended by Meunier [25]. In next subsection, we present a  $\mathbb{Z}_p$  version of Ky Fan's lemma which is called  $\mathbb{Z}_p$ -Tucker-Ky Fan lemma.

**3.2. New Generalizations of Tucker Lemma.** Before presenting our results, we need to introduce some functions having key roles in the paper. Throughout the paper, we are going to use these functions repeatedly. Let  $m$  be a positive integer. We remind that  $(\sigma_{p-2}^{p-1})^{*m}$  is a free simplicial  $\mathbb{Z}_p$ -complex with vertex set  $\mathbb{Z}_p \times [m]$ .

**The value function  $l(-)$ .** Let  $\tau \in (\sigma_{p-2}^{p-1})^{*m}$  be a simplex. For each  $\epsilon \in \mathbb{Z}_p$ , define  $\tau^\epsilon = \{(\epsilon, j) : (\epsilon, j) \in \tau\}$ . Moreover, define

$$l(\tau) = \max \left\{ \left| \bigcup_{\epsilon \in \mathbb{Z}_p} B^\epsilon \right| : \forall \epsilon \in \mathbb{Z}_p, B^\epsilon \subseteq \tau^\epsilon \text{ and } \forall \epsilon_1, \epsilon_2 \in \mathbb{Z}_p, |B^{\epsilon_1}| - |B^{\epsilon_2}| \leq 1 \right\}.$$

Note that if we set  $h(\tau) = \min_{\epsilon \in \mathbb{Z}_p} |\tau^\epsilon|$ , then

$$l(\tau) = p \cdot h(\tau) + |\{\epsilon \in \mathbb{Z}_p : |\tau^\epsilon| > h(\tau)\}|.$$

**The sign functions  $s(-)$  and  $s_0(-)$ .** For an  $a \in [m]$ , let  $W_a$  be the set of all simplices  $\tau \in (\sigma_{p-2}^{p-1})^{*m}$  such that  $|\tau^\epsilon| \in \{0, a\}$  for each  $\epsilon \in \mathbb{Z}_p$ . Let  $W = \bigcup_{a=1}^m W_a$ . Choose an arbitrary  $\mathbb{Z}_p$ -equivariant map  $s : W \rightarrow \mathbb{Z}_p$ . Also, consider an  $\mathbb{Z}_p$ -equivariant map  $s_0 : \sigma_{p-2}^{p-1} \rightarrow \mathbb{Z}_p$ . Note that since  $\mathbb{Z}_p$  acts freely on both  $\sigma_{p-2}^{p-1}$  and  $W$ , these maps can be easily built by choosing one representative in each orbit. It should be mentioned that both functions  $s(-)$  and  $s_0(-)$  are first introduced in [23].

Now, we are in a position to generalize Tucker-Ky Fan lemma to  $\mathbb{Z}_p$ -Tucker-Ky Fan lemma.

**Lemma 1.** ( $\mathbb{Z}_p$ -Tucker-Ky Fan lemma). *Let  $m, n, p$  and  $\alpha$  be nonnegative integers, where  $m, n \geq 1$ ,  $m \geq \alpha \geq 1$ , and  $p$  is prime. Let*

$$\begin{aligned} \lambda : (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\} &\longrightarrow \mathbb{Z}_p \times [m] \\ X &\longmapsto (\lambda_1(X), \lambda_2(X)) \end{aligned}$$

be a  $\mathbb{Z}_p$ -equivariant map satisfying the following conditions.

- For  $X_1 \subseteq X_2 \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\}$ , if  $\lambda_2(X_1) = \lambda_2(X_2) \leq \alpha$ , then  $\lambda_1(X_1) = \lambda_1(X_2)$ .
- For  $X_1 \subseteq X_2 \subseteq \dots \subseteq X_p \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\}$ , if  $\lambda_2(X_1) = \lambda_2(X_2) = \dots = \lambda_2(X_p) \geq \alpha + 1$ , then

$$|\{\lambda_1(X_1), \lambda_1(X_2), \dots, \lambda_1(X_p)\}| < p.$$

Then there is a chain

$$Z_1 \subset Z_2 \subset \dots \subset Z_{n-\alpha} \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\}$$

such that

- (1) for each  $i \in [n - \alpha]$ ,  $\lambda_2(Z_i) \geq \alpha + 1$ ,
- (2) for each  $i \neq j \in [n - \alpha]$ ,  $\lambda(Z_i) \neq \lambda(Z_j)$ , and
- (3) for each  $\epsilon \in \mathbb{Z}_p$ ,

$$\left\lfloor \frac{n - \alpha}{p} \right\rfloor \leq |\{j : \lambda_1(Z_j) = \epsilon\}| \leq \left\lceil \frac{n - \alpha}{p} \right\rceil.$$

In particular,  $n - \alpha \leq (p - 1)(m - \alpha)$ .

*Proof.* Note that the map  $\lambda$  can be considered as a simplicial  $\mathbb{Z}_p$ -map from  $\text{sd } \mathbb{Z}_p^{*n}$  to  $(\mathbb{Z}_p^{*\alpha}) * ((\sigma_{p-2}^{p-1})^{*(m-\alpha)})$ . Let  $K = \text{Im}(\lambda)$ . Note that each simplex in  $K$  can be represented in a unique form  $\sigma \cup \tau$  such that  $\sigma \in \mathbb{Z}_p^{*\alpha}$  and  $\tau \in (\sigma_{p-2}^{p-1})^{*m-\alpha}$ .

In view of definition of the function  $l(-)$  and the properties which  $\lambda$  satisfies in, to prove the assertion, it suffices to show that there is a simplex  $\sigma \cup \tau \in K$  such that  $l(\tau) \geq n - \alpha$ . For a contradiction, suppose that for each  $\sigma \cup \tau \in K$ , we have  $l(\tau) \leq n - \alpha - 1$ .

Define the map

$$\Gamma : \text{sd } K \longrightarrow \mathbb{Z}_p^{*(n-1)}$$

such that for each vertex  $\sigma \cup \tau \in V(\text{sd } K)$ ,

- if  $\tau = \emptyset$ , then  $\Gamma(\sigma \cup \tau) = (\epsilon, j)$ , where  $j$  is the maximum possible value such that  $(\epsilon, j) \in \sigma$ . Note that since  $\sigma \in \mathbb{Z}_p^{*\alpha}$ , there is only one  $\epsilon \in \mathbb{Z}_p$  for which the maximum is attained. Therefore, in this case, the function  $\Gamma$  is well-defined.
- if  $\tau \neq \emptyset$ . Define  $h(\tau) = \min_{\epsilon \in \mathbb{Z}_p} |\tau^\epsilon|$ .

- (i) If  $h(\tau) = 0$ , then define  $\bar{\tau} = \{\epsilon \in \mathbb{Z}_p : \tau^\epsilon = \emptyset\} \in \sigma_{p-2}^{p-1}$  and

$$\Gamma(\sigma \cup \tau) = (s_0(\bar{\tau}), \alpha + l(\tau)).$$

(ii) If  $h(\tau) > 0$ , then define  $\bar{\tau} = \bigcup_{\{\epsilon \in \mathbb{Z}_p : |\tau^\epsilon| = h(\tau)\}} \tau^\epsilon \in W$  and

$$\Gamma(\sigma \cup \tau) = (s(\bar{\tau}), \alpha + l(\tau)).$$

Now, we claim that  $\Gamma$  is a simplicial  $\mathbb{Z}_p$ -map from  $\text{sd } K$  to  $\mathbb{Z}_p^{*(n-1)}$ . It is clear that  $\Gamma$  is a  $\mathbb{Z}_p$ -equivariant map. For a contradiction, suppose that there are  $\sigma \cup \tau, \sigma' \cup \tau' \in \text{sd } K$  such that  $\sigma \subseteq \sigma'$ ,  $\tau \subseteq \tau'$ ,  $\Gamma(\sigma \cup \tau) = (\epsilon, \beta)$ , and  $\Gamma(\sigma' \cup \tau') = (\epsilon', \beta)$ , where  $\epsilon \neq \epsilon'$ . First note that in view of the definition of  $\Gamma$  and the assumption  $\Gamma(\sigma \cup \tau) = (\epsilon, \beta)$  and  $\Gamma(\sigma' \cup \tau') = (\epsilon', \beta)$ , the case  $\tau = \emptyset$  and  $\tau' \neq \emptyset$  is not possible. If  $\tau' = \emptyset$ , then  $\tau = \tau' = \emptyset$  and we should have  $(\epsilon, \beta), (\epsilon', \beta) \in \sigma' \in \mathbb{Z}_p^{*\alpha}$  which implies that  $\epsilon = \epsilon'$ , a contradiction. If  $\emptyset \neq \tau \subseteq \tau'$ , then in view of definition of  $\Gamma$ , we should have  $l(\tau) = l(\tau')$ . We consider three different cases.

(i) If  $h(\tau) = h(\tau') = 0$ , then

$$\epsilon = s_0(\{\epsilon \in \mathbb{Z}_p : \tau^\epsilon = \emptyset\}) \neq s_0(\{\epsilon \in \mathbb{Z}_p : \tau'^\epsilon = \emptyset\}) = \epsilon'.$$

Therefore,  $\{\epsilon \in \mathbb{Z}_p : \tau'^\epsilon = \emptyset\} \subsetneq \{\epsilon \in \mathbb{Z}_p : \tau^\epsilon = \emptyset\}$ . This implies that

$$l(\tau') = p - |\{\epsilon \in \mathbb{Z}_p : \tau'^\epsilon = \emptyset\}| > p - |\{\epsilon \in \mathbb{Z}_p : \tau^\epsilon = \emptyset\}| = l(\tau),$$

a contradiction.

(ii) If  $h(\tau) = 0$  and  $h(\tau') > 0$ . We should have  $l(\tau) \leq p - 1$  and  $l(\tau') \geq p$  which contradicts the fact that  $l(\tau) = l(\tau')$ .

(iii) If  $h(\tau) > 0$  and  $h(\tau') > 0$ . Note that

$$l(\tau) = p \cdot h(\tau) + |\{\epsilon \in \mathbb{Z}_p : |\tau^\epsilon| > h(\tau)\}| \text{ and } l(\tau') = p \cdot h(\tau') + |\{\epsilon \in \mathbb{Z}_p : |\tau'^\epsilon| > h(\tau')\}|.$$

For this case, two different sub-cases will be distinguished.

(a) If  $h(\tau) = h(\tau') = h$ , then

$$\epsilon = s\left(\bigcup_{\{\epsilon \in \mathbb{Z}_p : |\tau^\epsilon| = h\}} \tau^\epsilon\right) \neq s\left(\bigcup_{\{\epsilon \in \mathbb{Z}_p : |\tau'^\epsilon| = h\}} \tau'^\epsilon\right) = \epsilon'.$$

Clearly, it implies that

$$\bigcup_{\{\epsilon \in \mathbb{Z}_p : |\tau^\epsilon| = h\}} \tau^\epsilon \neq \bigcup_{\{\epsilon \in \mathbb{Z}_p : |\tau'^\epsilon| = h\}} \tau'^\epsilon.$$

Note that  $\tau \subseteq \tau'$  and  $h = \min_{\epsilon \in \mathbb{Z}_p} |\tau^\epsilon| = \min_{\epsilon \in \mathbb{Z}_p} |\tau'^\epsilon|$ . Therefore, we should have

$$\{\epsilon \in \mathbb{Z}_p : |\tau'^\epsilon| = h\} \subsetneq \{\epsilon \in \mathbb{Z}_p : |\tau^\epsilon| = h\}$$

and consequently  $l(\tau) < l(\tau')$  which is a contradiction.

(b) If  $h(\tau) < h(\tau')$ , then

$$l(\tau) \leq p \cdot h(\tau) + p - 1 < p \cdot (h(\tau) + 1) \leq l(\tau'),$$

a contradiction.

Therefore,  $\Gamma$  is a simplicial  $\mathbb{Z}_p$ -map from  $\text{sd } K$  to  $\mathbb{Z}_p^{*(n-1)}$ . Naturally,  $\lambda$  can be lifted to a simplicial  $\mathbb{Z}_p$ -map  $\bar{\lambda} : \text{sd}^2 \mathbb{Z}_p^{*n} \rightarrow \text{sd } K$ . Thus  $\Gamma \circ \bar{\lambda}$  is a simplicial  $\mathbb{Z}_p$ -map from  $\text{sd}^2 \mathbb{Z}_p^{*n}$  to  $\mathbb{Z}_p^{*(n-1)}$ . In view of Dold's theorem [10, 21], the dimension of  $\mathbb{Z}_p^{*(n-1)}$  should be strictly larger than the connectivity of  $\text{sd}^2 \mathbb{Z}_p^{*n}$ , that is  $n - 2 > n - 2$ , which is not possible.  $\square$

Lemma 1 provides a short simple proof of Meunier's colorful result for Kneser hypergraphs (next Theorem) as follows.

**Theorem C.** [23] *Let  $\mathcal{H}$  be a hypergraph and let  $p$  be a prime number. Then any proper coloring  $c : V(\text{KG}^p(\mathcal{H})) \rightarrow [C]$  ( $C$  arbitrary) must contain a colorful balanced complete  $p$ -uniform  $p$ -partite hypergraph with  $|V(\mathcal{H})| - \text{alt}_p(\mathcal{H})$  vertices.*

*Proof.* Consider a bijection  $\pi : [n] \rightarrow V(\mathcal{H})$  such that  $\text{alt}_p(\mathcal{H}, \pi) = \text{alt}_p(\mathcal{H})$ . We are going to define a map

$$\begin{aligned} \lambda : (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\} &\longrightarrow \mathbb{Z}_p \times [m] \\ X &\longmapsto (\lambda_1(X), \lambda_2(X)) \end{aligned}$$

satisfying the conditions of Lemma 1 and with parameters  $n = |V(\mathcal{H})|$ ,  $m = \text{alt}_p(\mathcal{H}) + C$ , and  $\alpha = \text{alt}_p(\mathcal{H})$ . Assume that  $2^{[n]}$  is equipped with a total ordering  $\preceq$ . For each  $X \in (\mathbb{Z}_p \cup \{0\})^n \setminus \{\mathbf{0}\}$ , define  $\lambda(X)$  as follows.

- If  $\text{alt}(X) \leq \text{alt}_p(\mathcal{H}, \pi)$ , then let  $\lambda_1(X)$  be the first nonzero coordinate of  $X$  and  $\lambda_2(X) = \text{alt}(X)$ .
- If  $\text{alt}(X) \geq \text{alt}_p(\mathcal{H}, \pi) + 1$ , then in view of the definition of  $\text{alt}_p(\mathcal{H}, \pi)$ , there is some  $\epsilon \in \mathbb{Z}_p$  such that  $E(\pi(X^\epsilon)) \neq \emptyset$ . Define

$$c(X) = \max \{c(e) : \exists \epsilon \in \mathbb{Z}_p \text{ such that } e \subseteq \pi(X^\epsilon)\}$$

and  $\lambda_2(X) = \text{alt}_p(\mathcal{H}, \pi) + c(X)$ . Choose  $\epsilon \in \mathbb{Z}_p$  such that there is at least one edge  $e \in \pi(X^\epsilon)$  with  $c(X) = c(e)$  and such that  $X^\epsilon$  is the maximum one having this property. By the maximum, we mean the maximum according to the total ordering  $\preceq$ . It is clear that  $\epsilon$  is defined uniquely. Now, let  $\lambda_1(X) = \epsilon$ .

One can check that  $\lambda$  satisfies the conditions of Lemma 1. Consider the chain  $Z_1 \subset Z_2 \subset \dots \subset Z_{n-\text{alt}_p(\mathcal{H}, \pi)}$  whose existence is ensured by Lemma 1. Note that for each  $i \in [n - \text{alt}_p(\mathcal{H}, \pi)]$ , we have  $\lambda_2(Z_i) > \text{alt}_p(\mathcal{H}, \pi)$ . Consequently,  $\lambda_2(Z_i) = \text{alt}_p(\mathcal{H}, \pi) + c(Z_i)$ . Let  $\lambda(Z_i) = (\epsilon_i, j_i)$ . Note that for each  $i$ , there is at least one edge  $e_{i, \epsilon_i} \subseteq \pi(Z_i^{\epsilon_i}) \subseteq \pi(Z_{n-\text{alt}_p(\mathcal{H}, \pi)}^{\epsilon_i})$  such that  $c(e_{i, \epsilon_i}) = j_i - \text{alt}_p(\mathcal{H}, \pi)$ . For each  $\epsilon \in \mathbb{Z}_p$ , define  $U_\epsilon = \{e_{i, \epsilon_i} : \epsilon_i = \epsilon\}$ . We have the following three properties for  $U_\epsilon$ 's.

- Since the chain  $Z_1 \subset Z_2 \subset \dots \subset Z_{n-\text{alt}_p(\mathcal{H}, \pi)}$  is satisfying Condition 3 of Lemma 1, we have  $\left\lfloor \frac{n-\text{alt}_p(\mathcal{H}, \pi)}{p} \right\rfloor \leq |U_\epsilon| \leq \left\lceil \frac{n-\text{alt}_p(\mathcal{H}, \pi)}{p} \right\rceil$ .
- The edges in  $U_\epsilon$  get distinct colors. If there are two edges  $e_{i, \epsilon}$  and  $e_{i', \epsilon}$  in  $U_\epsilon$  such that  $c(e_{i, \epsilon}) = c(e_{i', \epsilon})$ , then  $\lambda(Z_i) = \lambda(Z_{i'})$  which is not possible.
- If  $\epsilon \neq \epsilon'$ , then for each  $e \in U_\epsilon$  and  $f \in U_{\epsilon'}$ , we have  $e \cap f = \emptyset$ . It is clear because  $e \subseteq \pi(Z_{n-\text{alt}_p(\mathcal{H}, \pi)}^\epsilon)$ ,  $f \subseteq \pi(Z_{n-\text{alt}_p(\mathcal{H}, \pi)}^{\epsilon'})$ , and

$$\pi(Z_{n-\text{alt}_p(\mathcal{H}, \pi)}^\epsilon) \cap \pi(Z_{n-\text{alt}_p(\mathcal{H}, \pi)}^{\epsilon'}) = \emptyset.$$

Now, it is clear that the subhypergraph  $\text{KG}^p(\mathcal{H})[U_{\omega_1}, \dots, U_{\omega_p}]$  is the desired subhypergraph.  $\square$

The proof of next lemma is similar to the proof of Lemma 1.

**Lemma 2.** *Let  $C$  be a free simplicial  $\mathbb{Z}_p$ -complex such that  $\text{ind}_{\mathbb{Z}_p}(C) \geq t$  and let  $\lambda : C \rightarrow (\sigma_{p-2}^{p-1})^{*m}$  be a simplicial  $\mathbb{Z}_p$ -map. Then there is at least one  $t$ -dimensional simplex  $\sigma \in C$  such that  $\tau = \lambda(\sigma)$  is a  $t$ -dimensional simplex and for each  $\epsilon \in \mathbb{Z}_p$ , we have  $\lfloor \frac{t+1}{p} \rfloor \leq |\tau^\epsilon| \leq \lceil \frac{t+1}{p} \rceil$ .*

*Proof.* For simplicity of notation, let  $K = \text{Im}(\lambda)$ . Clearly, to prove the assertion, it is enough to show that there is a  $t$ -dimensional simplex  $\tau \in K$  such that  $l(\tau) \geq t$ . Suppose, contrary to the assertion, that there is no such a  $t$ -dimensional simplex. Therefore, for each simplex  $\tau$  of  $K$ , we have  $l(\tau) \leq t$ . For each vertex  $\tau \in V(\text{sd } K)$ , set  $h(\tau) = \min_{\epsilon \in \mathbb{Z}_p} |\tau^\epsilon|$ .

Let  $\Gamma : \text{sd } K \rightarrow \mathbb{Z}_p^{*t}$  be a map such that for each vertex  $\tau$  of  $\text{sd } K$ ,  $\Gamma(\tau)$  is defined as follows.

(i) If  $h(\tau) = 0$ , then define  $\bar{\tau} = \{\epsilon \in \mathbb{Z}_p : \tau^\epsilon = \emptyset\} \in \sigma_{p-2}^{p-1}$  and

$$\Gamma(\sigma \cup \tau) = (s_0(\bar{\tau}), l(\tau)).$$

(ii) If  $h(\tau) > 0$ , then define  $\bar{\tau} = \bigcup_{\{\epsilon \in \mathbb{Z}_p : |\tau^\epsilon| = h(\tau)\}} \tau^\epsilon \in W$  and

$$\Gamma(\sigma \cup \tau) = (s(\bar{\tau}), l(\tau)).$$

Similar to the proof of Lemma 1,  $\Gamma \circ \bar{\lambda} : \text{sd } C \longrightarrow \mathbb{Z}_p^{*t}$  is a simplicial  $\mathbb{Z}_p$ -map. This implies that  $\text{ind}_{\mathbb{Z}_p}(C) \leq t - 1$  which is not possible.  $\square$

Next proposition is an extension of Theorem B. However, we lose some properties by this extension.

**Proposition 1.** *Let  $P$  be a free  $\mathbb{Z}_p$ -poset and*

$$\begin{aligned} \psi : P &\longrightarrow Q_{s,p} \\ p &\longmapsto (\psi_1(p), \psi_2(p)) \end{aligned}$$

*be an order preserving  $\mathbb{Z}_p$ -map. Then  $P$  contains a chain  $p_1 \prec_P \cdots \prec_P p_k$  such that*

- $k = \text{ind}_{\mathbb{Z}_p}(\Delta P) + 1$ ,
- *for each  $i \in [k - 1]$ ,  $\psi_2(p_i) < \psi_2(p_{i+1})$ , and*
- *for each  $\epsilon \in \mathbb{Z}_p$ ,*

$$\left\lfloor \frac{k}{p} \right\rfloor \leq |\{j : \psi_1(p_j) = \epsilon\}| \leq \left\lceil \frac{k}{p} \right\rceil.$$

*Proof.* Note  $\psi$  can be considered as a simplicial  $\mathbb{Z}_p$ -map from  $\Delta P$  to  $\mathbb{Z}_p^{*n} \subseteq (\sigma_{p-2}^{p-1})^{*n}$ . Now, in view of Lemma 2, we have the assertion.  $\square$

Note that, for  $p = 2$ , since  $\text{Xind}(P) \geq \text{ind}(\Delta P)$ , Theorem B is better than proposition 1. However, we cannot prove that proposition 1 is valid if we replace  $\text{ind}(\Delta P)$  by  $\text{Xind}(P)$ .

In an unpublished paper, Meunier [24] introduced a generalization of Tuckey-Ky Fan lemma. He presented a version of  $\mathbb{Z}_q$ -Fan lemma which is valid for each odd integer  $q \geq 3$ . To be more specific, he proved that if  $q$  is an odd positive integer and  $\lambda : V(T) \longrightarrow \mathbb{Z}_q \times [m]$  is a  $\mathbb{Z}_q$ -equivariant labeling of an  $\mathbb{Z}_q$ -equivariant triangulation of a  $(d - 1)$ -connected free  $\mathbb{Z}_q$ -spaces  $T$ , then there is at least one simplex in  $T$  whose vertices are labelled with labels  $(\epsilon_0, j_0), (\epsilon_1, j_1), \dots, (\epsilon_n, j_n)$ , where  $\epsilon_i \neq \epsilon_{i+1}$  and  $j_i < j_{i+1}$  for all  $i \in \{0, 1, \dots, n - 1\}$ . Also, he asked the question if the result is true for even value of  $q$ . This question received a positive answer owing to the work of B. Hanke et al. [16]. In both mentioned works, the proofs of  $\mathbb{Z}_q$ -Fan lemma are built in involved construction. Here, we take the opportunity of this paper to propose the following generalization of this result with a short simple proof because we are using similar techniques in the paper.

**Lemma 3.** ( $\mathbb{G}$ -Fan lemma). *Let  $\mathbb{G}$  be a nontrivial finite group and let  $T$  be a free  $\mathbb{G}$ -simplicial complex such that  $\text{ind}_{\mathbb{G}}(T) = n$ . Assume that  $\lambda : V(T) \longrightarrow \mathbb{G} \times [m]$  be a  $\mathbb{G}$ -equivariant labeling such that there is no edge in  $T$  whose vertices are labelled with  $(g, j)$  and  $(g', j)$  with  $g \neq g'$  and  $j \in [m]$ . Then there is at least one simplicial complex in  $T$  whose vertices are labelled with labels  $(g_0, j_0), (g_1, j_1), \dots, (g_n, j_n)$ , where  $g_i \neq g_{i+1}$  and  $j_i < j_{i+1}$  for all  $i \in \{0, 1, \dots, n - 1\}$ . In particular,  $m \geq n + 1$ .*

*Proof.* Clearly, the map  $\lambda$  can be considered as a  $\mathbb{G}$ -simplicial map from  $T$  to  $\mathbb{G}^{*m}$ . Naturally, each nonempty simplex  $\sigma \in \mathbb{G}^{*m}$  can be identified with a vector  $X = (x_1, x_2, \dots, x_m) \in (\mathbb{G} \cup \{0\})^n \setminus \{0\}$ .



To prove the assertion, it is enough to show that there is a simplex  $\sigma \in T$  such that  $\text{alt}(\lambda(\sigma)) \geq n+1$ . For a contradiction, suppose that, for each simplex  $\sigma \in T$ , we have  $\text{alt}(\lambda(\sigma)) \leq n$ . Define

$$\begin{aligned} \Gamma : V(\text{sd } T) &\longrightarrow \mathbb{G} \times [n] \\ \sigma &\longmapsto (g, \text{alt}(\lambda(\sigma))), \end{aligned}$$

where  $g$  is the first nonzero coordinate of the vector  $\lambda(\sigma) \in (\mathbb{G} \cup \{0\})^n \setminus \{\mathbf{0}\}$ . One can check that  $\Gamma$  is a simplicial  $\mathbb{G}$ -map from  $\text{sd } T$  to  $\mathbb{G}^{*n}$ . Note  $\mathbb{G}^{*n}$  is an  $E_{n-1}\mathbb{G}$  space. Consequently,  $\text{ind}_{\mathbb{G}}(\mathbb{G}^{*n}) = n-1$ . This implies that  $\text{ind}_{\mathbb{G}}(T) \leq n-1$  which is a contradiction.  $\square$

**3.3. Hierarchy of Indices.** The aim of this subsection is introducing some tools for the proof of Theorem 2.

Let  $n, \alpha$ , and  $p$  be integers where  $n \geq 1$ ,  $n \geq \alpha \geq 0$ , and  $p$  is prime. Define

$$\Sigma_p(n, \alpha) = \Delta \{X \in (\mathbb{Z}_p \cup \{0\})^n : \text{alt}(X) \geq \alpha + 1\}.$$

Note that  $\Sigma_p(n, \alpha)$  is a free simplicial  $\mathbb{Z}_p$ -complex with the vertex set

$$\{X \in (\mathbb{Z}_p \cup \{0\})^n : \text{alt}(X) \geq \alpha + 1\}.$$

**Lemma 4.** *Let  $n, \alpha$ , and  $p$  be integers where  $n \geq 1$ ,  $n \geq \alpha \geq 0$ , and  $p$  is prime. Then*

$$\text{ind}_{\mathbb{Z}_p}(\Sigma_p(n, \alpha)) \geq n - \alpha - 1.$$

*Proof.* Define

$$\begin{aligned} \lambda : \text{sd } \mathbb{Z}_p^{*n} &\longrightarrow (\mathbb{Z}_p^{*\alpha}) * (\Sigma_p(n, \alpha)) \\ X &\longmapsto \begin{cases} (\epsilon, \text{alt}(X)) & \text{if } \text{alt}(X) \leq \alpha \\ X & \text{if } \text{alt}(X) \geq \alpha + 1, \end{cases} \end{aligned}$$

where  $\epsilon$  is the first nonzero term of  $X$ . Clearly, the map  $\lambda$  is a simplicial  $\mathbb{Z}_p$ -map. Therefore,

$$\begin{aligned} n - 1 = \text{ind}_{\mathbb{Z}_p}(\text{sd } \mathbb{Z}_p^{*n}) &\leq \text{ind}_{\mathbb{Z}_p}(\mathbb{Z}_p^{*\alpha} * \Sigma_p(n, \alpha)) \\ &\leq \text{ind}_{\mathbb{Z}_p}(\mathbb{Z}_p^{*\alpha}) + \text{ind}_{\mathbb{Z}_p}(\Sigma_p(n, \alpha)) + 1 \\ &\leq \alpha + \text{ind}_{\mathbb{Z}_p}(\Sigma_p(n, \alpha)) \end{aligned}$$

which completes the proof.  $\square$

**Proposition 2.** *Let  $\mathcal{H}$  be a hypergraph. For any integer  $r \geq 2$  and any prime number  $p \geq r$ , we have*

$$\text{ind}_{\mathbb{Z}_p}(\text{B}_0(\text{KG}^r(\mathcal{H}), \mathbb{Z}_p)) + 1 \geq |V(\mathcal{H})| - \text{alt}_p(\mathcal{H}).$$

*Proof.* For convenience, let  $|V(\mathcal{H})| = n$  and  $\alpha = n - \text{alt}_p(\mathcal{H})$ . Let  $\pi : [n] \longrightarrow V(\mathcal{H})$  be the bijection such that  $\text{alt}_p(\mathcal{H}, \pi) = \text{alt}_p(\mathcal{H})$ . Define

$$\begin{aligned} \lambda : \Sigma_p(n, \alpha) &\longrightarrow \text{sd } \text{B}_0(\text{KG}^r(\mathcal{H}), \mathbb{Z}_p) \\ X &\longmapsto \{\omega^1\} \times U_1 \cup \cdots \cup \{\omega^p\} \times U_p, \end{aligned}$$

where  $U_i = \{e \in E(\mathcal{H}) : e \subseteq \pi(X^{\omega^i})\}$ . One can see that  $\lambda$  is a simplicial  $\mathbb{Z}_p$ -map. Consequently,

$$\text{ind}_{\mathbb{Z}_p}(\text{B}_0(\text{KG}^r(\mathcal{H}), \mathbb{Z}_p)) \geq \text{ind}_{\mathbb{Z}_p}(\Sigma_p(n, \alpha)) \geq n - \text{alt}_p(\mathcal{H}) - 1.$$

$\square$

**Proposition 3.** *Let  $\mathcal{H}$  be an  $r$ -uniform hypergraph and  $p \geq r$  be a prime number. Then*

$$\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^r, \mathcal{H})) + p \geq \text{ind}_{\mathbb{Z}_p}(\Delta \text{Hom}(K_p^r, \mathcal{H})) + p \geq \text{ind}_{\mathbb{Z}_p}(\text{B}_0(\mathcal{H}, \mathbb{Z}_p)) + 1.$$

*Proof.* Since already we know  $\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^r, \mathcal{H})) \geq \text{ind}_{\mathbb{Z}_p}(\Delta\text{Hom}(K_p^r, \mathcal{H}))$ , to prove the assertion, it is enough to show that  $\text{ind}_{\mathbb{Z}_p}(\Delta\text{Hom}(K_p^r, \mathcal{H})) + p \geq \text{ind}_{\mathbb{Z}_p}(B_0(\mathcal{H}, \mathbb{Z}_p)) + 1$ . To this end, define

$$\lambda : \text{sd } B_0(\mathcal{H}, \mathbb{Z}_p) \longrightarrow \left( \text{sd } \sigma_{p-2}^{p-1} \right) * (\Delta\text{Hom}(K_p^r, \mathcal{H}))$$

such that for each vertex  $\tau = \bigcup_{i=1}^p (\{\omega^i\} \times U_i)$  of  $\text{sd } B_0(\mathcal{H}, \mathbb{Z}_p)$ ,  $\lambda(\tau)$  is defined as follows.

- If  $U_i \neq \emptyset$  for each  $i \in [p]$ , then  $\lambda(\tau) = \tau$ .
- If  $U_i = \emptyset$  for some  $i \in [p]$ , then

$$\lambda(\tau) = \{\omega^i \in \mathbb{Z}_p : U_i = \emptyset\}.$$

One can check that the map  $\lambda$  is a simplicial  $\mathbb{Z}_p$ -map. Also, since  $\sigma_{p-2}^{p-1}$  is a free simplicial  $\mathbb{Z}_p$ -complex of dimension  $p-2$ , we have  $\text{ind}_{\mathbb{Z}_p}(\sigma_{p-2}^{p-1}) \leq p-2$  (see properties of the  $\mathbb{G}$ -index in Section 2). This implies that

$$\begin{aligned} \text{ind}_{\mathbb{Z}_p}(B_0(\mathcal{H}, \mathbb{Z}_p)) &\leq \text{ind}_{\mathbb{Z}_p} \left( \left( \text{sd } \sigma_{p-2}^{p-1} \right) * (\Delta\text{Hom}(K_p^r, \mathcal{H})) \right) \\ &\leq \text{ind}_{\mathbb{Z}_p}(\sigma_{p-2}^{p-1}) + \text{ind}_{\mathbb{Z}_p}(\Delta\text{Hom}(K_p^r, \mathcal{H})) + 1 \\ &\leq p-1 + \text{ind}_{\mathbb{Z}_p}(\Delta\text{Hom}(K_p^r, \mathcal{H})) \end{aligned}$$

which completes the proof.  $\square$

#### 4. Proofs of Theorem 1 and Theorem 2

Now, we are ready to prove Theorem 1 and Theorem 2.

**Proof of Theorem 1: Part (i).** For convenience, let  $\text{ind}_{\mathbb{Z}_p}(B_0(\mathcal{H}, \mathbb{Z}_p)) = t$ . Note that

$$\begin{aligned} \Gamma : \mathbb{Z}_p \times V(\mathcal{H}) &\longrightarrow \mathbb{Z}_p \times [C] \\ (\epsilon, v) &\longmapsto (\epsilon, c(v)) \end{aligned}$$

is a simplicial  $\mathbb{Z}_p$ -map from  $B_0(\mathcal{H}, \mathbb{Z}_p)$  to  $(\sigma_{r-2}^{p-1})^{*C}$ . Therefore, in view of Lemma 2, there is a  $t$ -dimensional simplex  $\tau \in \text{im}(\Gamma)$  such that, for each  $\epsilon \in \mathbb{Z}_p$ , we have  $\lfloor \frac{t+1}{p} \rfloor \leq |\tau^\epsilon| \leq \lceil \frac{t+1}{p} \rceil$ . Let

$\bigcup_{i=1}^p (\{\omega^i\} \times U_i)$  be the minimal simplex in  $\Gamma^{-1}(\tau)$ . One can see that  $\mathcal{H}[U_1, \dots, U_p]$  is the desired subhypergraph. Moreover, since every color can be appeared in at most  $r-1$  number of  $U_i$ 's, we have

$$C \geq \frac{\text{ind}_p(B_0(\mathcal{H}, \mathbb{Z}_p)) + 1}{r-1}.$$

**Part (ii).** For convenience, let  $\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^r, \mathcal{H})) = t$ . Define the map

$$\lambda : \text{Hom}(K_p^r, \mathcal{H}) \longrightarrow \text{sd}(\sigma_{r-2}^{p-1})^{*C}$$

such that for each  $(U_1, \dots, U_p) \in \text{Hom}(K_p^r, \mathcal{H})$ ,

$$\lambda(U_1, \dots, U_p) = \{\omega^1\} \times c(U_1) \cup \dots \cup \{\omega^p\} \times c(U_p).$$

**Claim.** There is a  $p$ -tuple  $(U_1, \dots, U_p) \in \text{Hom}(K_p^r, \mathcal{H})$  such that for  $\tau = \lambda(U_1, \dots, U_p)$ , we have  $l(\tau) \geq \text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^r, \mathcal{H})) + p$ .

**Proof of Claim.** Suppose, contrary to the claim, that for each  $\tau \in \text{Im}(\lambda)$ , we have  $l(\tau) \leq t + p - 1$ . Note that  $\text{sd}(\sigma_{r-2}^{p-1})^{*C}$  can be considered as a free  $\mathbb{Z}_p$ -poset ordered by inclusion. One can readily

check that  $\lambda$  is an order-preserving  $\mathbb{Z}_p$ -map. Clearly, for each  $\tau \in \text{Im}(\lambda)$ , we have  $h(\tau) = \min_{\epsilon \in \mathbb{Z}_p} |\tau^\epsilon| \geq 1$  and consequently,  $l(\tau) \geq p$ . Now, define

$$\bar{\tau} = \bigcup_{\{\epsilon \in \mathbb{Z}_p : |\tau^\epsilon| = h(\tau)\}} \tau^\epsilon \in W \quad \text{and} \quad \Gamma(\tau) = (s(\bar{\tau}), l(\tau) - p + 1).$$

One can see that the map  $\Gamma : \text{im}(\lambda) \rightarrow Q_{t-1,p}$  is an order-preserving  $\mathbb{Z}_p$ -map. Therefore,

$$\Gamma \circ \lambda : \text{Hom}(K_p^r, \mathcal{H}) \rightarrow Q_{t-1,p}$$

is an order-preserving  $\mathbb{Z}_p$ -map, which contradicts the fact that  $\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^r, \mathcal{H})) = t$ .  $\square$

Now, let  $(U_1, \dots, U_p)$  be a minimal  $p$ -tuple in  $\text{Hom}(K_p^r, \mathcal{H})$  such that for

$$\tau = \lambda(U_1, \dots, U_p) = \{\omega^1\} \times c(U_1) \cup \dots \cup \{\omega^p\} \times c(U_p),$$

we have  $l(\tau) = t + p$ . One can check that  $\mathcal{H}[U_1, \dots, U_p]$  is the desired complete  $r$ -uniform  $p$ -partite subhypergraph. Similar to the proof of Part (i), since every color can be appeared in at most  $r - 1$  number of  $U_i$ 's, we have

$$C \geq \frac{\text{Xind}_{\mathbb{Z}_p}(\text{Hom}(K_p^r, \mathcal{H})) + p}{r - 1}.$$

$\square$

**Proof of Theorem 2.** It is simple to prove that  $|V(\mathcal{F})| - \text{alt}_p(\mathcal{F}) \geq \text{cd}_p(\mathcal{F})$  for any hypergraph  $\mathcal{F}$ . Therefore, the proof follows by Proposition 2 and Proposition 3.  $\square$

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